# The Integration by Parts Method and Going in Circles

This article assumes you know the integration by parts technique. If you don't, you might want to brush up on the method before continuing.

For some integration problems, you have to use the integration by parts method more than once because the first run through the process takes you only part way to the answer. Here's an example. Find $\int x^2 e^x dx$.

1. **Pick your *u*.**

   You may have your own method for deciding on what chunk of the integrand will become your $u$. The *LIATE* method is a good one.

   **The *LIATE* mnemonic.** Herbert E. Kasube came up with the acronym *LIATE* to help you choose your $u$ (you can check out Herb's article in the *American Mathematical Monthly* 90, 1983 issue):

   | | | |
   |---|---|---|
   | *L* | Logarithmic | (like $\log(x)$) |
   | *I* | Inverse trigonometric | (like $\arctan(x)$) |
   | *A* | Algebraic | (like $5x^2+3$) |
   | *T* | Trigonometric | (like $\cos(x)$) |
   | *E* | Exponential | (like $10^x$) |

   To pick your $u$ chunk, go down this list in order; the first type of function on this list that appears in the integrand is the $u$. Once you've selected your $u$, everything else is automatically the $dv$ chunk.

   $x^2 e^x dx$ contains an algebraic function, $x^2$, and an exponential function, $e^x$ (It's an exponential function because there's an $x$ in the exponent). The first on the *LIATE* list is the *Algebraic* function $x^2$, so that's your $u$.

2. **Differentiate to get *du*, and integrate to get *v* (you ignore the *C*).**

   $$u = x^2 \qquad dv = e^x dx$$
   $$\frac{du}{dx} = 2x \qquad \int dv = \int e^x dx$$
   $$du = 2x\,dx \qquad v = e^x$$

3. **Use the integration-by-parts formula.**

$$\int u\,dv = uv - \int v\,du$$

$$\int x^2 e^x dx = x^2 e^x - \int e^x \cdot 2x\,dx$$
$$= x^2 e^x - 2\int x e^x dx$$

You end up with another integral, $\int x e^x dx$, that can't be done by any of the simple methods — reverse rules, guess and check, and substitution. But note that the power of $x$ has been reduced by one, so you've made some progress. If you use integration by parts again for $\int x e^x dx$, the $x$ disappears entirely and you're done.

4. **Integrate by parts again.**

You can do most of this one on your own. Here's the final step:

$$\int x e^x dx = x e^x - \int e^x dx$$
$$= x e^x - e^x + C$$

5. **Take the result from Step 4 and substitute it for the $\int x e^x dx$ in the answer from Step 3 to produce the whole enchilada.**

$$\int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x + C)$$
$$= x^2 e^x - 2x e^x + 2e^x - 2C$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$

## Going around in circles

Sometimes if you use integration by parts twice, you get back to where you started from — which, unlike getting lost, is *not* a waste of time. Integrate $\int e^x \cos(x)dx$ and be enlightened.

Your $u$ is $\cos(x)$ (it's *T* in *LIATE*), and $e^x$ is your $dv$. Now fast forward to the formula step:

$$\int u\,dv = uv - \int v\,du$$
$$\int e^x \cos(x)\,dx = e^x \cos(x) - \int e^x(-\sin(x))\,dx$$
$$= e^x \cos(x) + \int e^x \sin(x)\,dx$$

Integrating by parts again for $\int e^x \sin(x)dx$ gives you

$$\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx$$

And you're back to where you started from: $\int e^x \cos(x)dx$. No worries. First, substitute the right side of the above equation for the $\int e^x \sin(x)dx$ from the original solution:

$$\int e^x \cos(x)\,dx = e^x \cos(x) + \int e^x \sin(x)\,dx$$
$$\int e^x \cos(x)\,dx = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\,dx$$

You can now solve this equation for the integral $\int e^x \cos(x)dx$. Use $I$ in place of that integral to make this messy equation easier on the eyes:

$$I = e^x \cos(x) + e^x \sin(x) - I$$

Add $I$ to both sides:

$$2I = e^x \cos(x) + e^x \sin(x)$$

Multiply both sides by $\frac{1}{2}$.

$$I = \frac{1}{2}(e^x \cos(x) + e^x \sin(x))$$
$$= \frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x)$$

Finally, put the $\int e^x \cos(x)dx$ back in for the $I$, and don't forget the $C$:

$$\int e^x \cos(x)\,dx = \frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C$$

That's a wrap.