Trig Integrals Containing Sines and Cosines, Secants and Tangents, or Cosecants and Cotangents

In this article, you integrate powers of the six trigonometric functions, like $\int \sin^3(x)\,dx$ and $\int \sec^4(x)\,dx$, and products or quotients of different trig functions, like $\int \sin^2(x)\cos^3(x)\,dx$ and $\int \frac{\csc^2(x)}{\cot(x)}\,dx$.

To use the following techniques, you must either have an integrand that contains just one of the six trig functions like $\int \csc^3(x)\,dx$ or a certain pairing of trig functions, like $\int \sin^2(x)\cos(x)\,dx$. If the integrand has two trig functions, the two must be one of these three pairs: sine with cosine, secant with tangent, or cosecant with cotangent. If you have an integrand containing something other than one of these three pairs, you can easily convert the problem into one of these pairs by using trig identities like $\sin(x) = \frac{1}{\csc(x)}$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$. For instance,

$$\begin{aligned}
&\int \sin^2(x)\sec(x)\tan(x)\,dx \\
&= \int \sin^2(x)\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\,dx \\
&= \int \frac{\sin^3(x)}{\cos^2(x)}\,dx
\end{aligned}$$

After doing any needed conversions, you want to get one of the following three cases:

$$\int \sin^m(x)\ \cos^n(x)\,dx$$

$$\int \sec^m(x)\ \tan^n(x)\,dx$$

$$\int \csc^m(x)\ \cot^n(x)\,dx$$

where either m or n (or both) is a positive integer.

The basic idea with most of the following trig integrals is to organize the integrand so that you can make a handy u-substitution and then integrate with the reverse power rule.

Integrals containing sines and cosines

This section covers integrals containing — can you guess? — sines and cosines.

Case 1: The power of sine is odd and positive

If the power of sine is odd and positive, lop off one sine factor and put it to the right of the rest of the expression, convert the remaining sine factors to cosines with the Pythagorean identity, and then integrate with the substitution method where $u = \cos(x)$.

The Pythagorean identity. The Pythagorean identity tells you that, for any angle x, $\sin^2(x) + \cos^2(x) = 1$. And thus $\sin^2(x) = 1 - \cos^2(x)$ and $\cos^2(x) = 1 - \sin^2(x)$.

Now integrate $\int \sin^3(x)\cos^4(x)\,dx$.

1. **Lop off one sine factor and move it to the right.**

$$\int \sin^3(x)\cos^4(x)\,dx = \int \sin^2(x)\cos^4(x)\ \sin(x)\,dx$$

2. **Convert the remaining sines to cosines using the Pythagorean identity and simplify.**

$$\begin{aligned} &\int \sin^2(x)\ \cos^4(x)\ \sin(x)\,dx \\ &= \int \left(1-\cos^2(x)\right)\ \cos^4(x)\ \sin(x)\,dx \\ &= \int \left(\cos^4(x)-\cos^6(x)\right)\ \sin(x)\,dx \end{aligned}$$

3. **Integrate with the substitution method, where $u = \cos(x)$.**

$$\begin{aligned} u &= \cos(x) \\ \frac{du}{dx} &= -\sin(x) \\ du &= -\sin(x)dx \end{aligned}$$

TIP

Shortcut for the *u*-substitution integration method. You can save a little time in all substitution problems by just solving for du — as was done immediately above — and not bothering to solve for dx. You then tweak the expression inside the integral so that it contains the thing du equals and then compensate for that tweaking by adding something outside the integral. In the current problem, du equals $-\sin(x)dx$. The integral contains a $\sin(x)dx$, so you multiply it by -1 to turn it into $-\sin(x)dx$ and then compensate for that -1 by multiplying the whole integral by -1. This is a wash because -1 times -1 equals 1. This may not sound like much of a shortcut, but it's a good time saver once you get used to it.

So tweak your integral:

$$\begin{aligned} &\int (\cos^4(x)-\cos^6(x))(\sin(x)\,dx) \\ &= -\int (\cos^4(x)-\cos^6(x))\,(-\sin(x)\,dx) \end{aligned}$$

Now do the *u*-substitution and solve by the reverse power rule:

$$\begin{aligned} &= -\int (u^4-u^6)\,du \\ &= -\frac{1}{5}u^5+\frac{1}{7}u^7+C \\ &= \frac{1}{7}\cos^7(x)-\frac{1}{5}\cos^5(x)+C \end{aligned}$$

Case 2: The power of cosine is odd and positive

This problem works exactly like Case 1, except that the roles of sine and cosine are reversed. Find $\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}\,dx$.

1. **Lop off one cosine factor and move it to the right.**

$$\begin{aligned}\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}\,dx &= \int \cos^3(x)\left(\sin^{-1/2}(x)\right)dx\\ &= \int \cos^2(x)\left(\sin^{-1/2}(x)\right)\,\cos(x)\,dx\end{aligned}$$

2. **Convert the remaining cosines to sines with the Pythagorean identity and simplify.**

$$\begin{aligned}&\int \cos^2(x)(\sin^{-1/2}(x))\,\cos(x)\,dx\\ &= \int (1-\sin^2(x))(\sin^{-1/2}(x))\,\cos(x)\,dx\\ &= \int (\sin^{-1/2}(x)-\sin^{3/2}(x))\,\cos(x)\,dx\end{aligned}$$

3. **Integrate with substitution, where $u = \sin(x)$.**

$$\begin{aligned}u &= \sin(x)\\ \frac{du}{dx} &= \cos(x)\\ du &= \cos(x)dx\end{aligned}$$

Now substitute:

$$= \int (u^{-1/2} - u^{3/2})\,du$$

And finish integrating as in Case 1.

Case 3: The powers of both sine and cosine are even and nonnegative

Here you convert the integrand into odd powers of cosines by using the following trig identities:

REMEMBER

Two handy trig identities.

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \text{ and } \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Then you finish the problem as in Case 2. Here's an example:

$$
\begin{aligned}
&\int \sin^4(x)\cos^2(x)\,dx \\
&= \int (\sin^2(x))^2 \cos^2(x)\,dx \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right) dx \\
&= \frac{1}{8}\int (1-\cos(2x)-\cos^2(2x)+\cos^3(2x))\,dx \\
&= \frac{1}{8}\int 1\,dx - \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{8}\int \cos^2(2x)\,dx + \frac{1}{8}\int \cos^3(2x)\,dx
\end{aligned}
$$

The first in this string of integrals is a no-brainer; the second is a simple reverse rule with a little tweak for the 2; you do the third integral by using the $\cos^2(x)$ identity a second time; and the fourth integral is handled by following the steps in Case 2. Do it. Your final answer should be

$$\frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C$$

A veritable cake walk.

Don't forget your trig identities. If you get a sine-cosine problem that doesn't fit any of the three cases discussed above, try using a trig identity like $\sin^2(x)+\cos^2(x)=1$ or $\cos^2(x)=\dfrac{1+\cos(2x)}{2}$ to convert the integral into one you can handle.

For example, $\displaystyle\int \frac{\sin^4(x)}{\cos^2(x)}\,dx$ doesn't fit any of the three sine-cosine cases, but you can use the Pythagorean identity to convert it to $\displaystyle\int \frac{(1-\cos^2(x))^2}{\cos^2(x)}\,dx = \int \frac{1-2\cos^2(x)+\cos^4}{\cos^2(x)}\,dx$. This splits up into $\displaystyle\int \sec^2(x)\,dx - \int 2\,dx + \int \cos^2(x)\,dx$, and the rest is easy. Try it. And see whether you can differentiate your result and arrive back at the original problem. *Spoiler alert* — here's the answer: $\tan(x) - \dfrac{3x}{2} + \dfrac{\sin(2x)}{4} + C$

Integrals containing secants and tangents

Ready for a shock? This section is about integrals containing secants and tangents.

Case 1: The power of tangent is odd and positive

Integrate $\int \sqrt{\sec(x)}\tan^3(x)\,dx$.

1. **Lop off a secant-tangent factor and move it to the right.**

   First, rewrite the problem: $\int \sqrt{\sec(x)}\tan^3(x)\,dx = \int \sec^{1/2}(x)\tan^3(x)\,dx$

   Now, taking a secant-tangent factor out of $\sec^{1/2}(x)\tan^3(x)$ may seem like trying to squeeze blood from a turnip because $\sec^{1/2}(x)$ has a power less than $\sec^1(x)$, but it works:

   $$\int \sec^{1/2}(x)\tan^3(x)\,dx = \int \left(\sec^{-1/2}(x)\tan^2(x)\right)\ \sec(x)\ \tan(x)\,dx$$

2. **Convert the remaining tangents to secants with the tangent-secant version of the Pythagorean identity.**

   Remembering the other two versions of the Pythagorean identity. An easy way to remember the tangent-secant version of the Pythagorean identity is to start with the sine-cosine version, $\sin^2(x)+\cos^2(x)=1$, and divide both sides of this equation by $\cos^2(x)$. This produces $\tan^2(x)+1=\sec^2(x)$. To produce the cotangent-cosecant version, divide both sides of $\sin^2(x)+\cos^2(x)=1$, by $\sin^2(x)$. The result is $1+\cot^2(x)=\csc^2(x)$.

   The Pythagorean identity is $\tan^2(x)+1=\sec^2(x)$, and thus $\tan^2(x)=\sec^2(x)-1$. Now make the switch.

   $$\int \left(\sec^{-1/2}(x)\tan^2(x)\right)\ \sec(x)\ \tan(x)\,dx$$
   $$= \int \left(\sec^{-1/2}(x)(\sec^2(x)-1)\right)\ \sec(x)\ \tan(x)\,dx$$
   $$= \int \left(\sec^{3/2}(x)-\sec^{-1/2}(x)\right)\ \sec(x)\ \tan(x)\,dx$$

3. **Solve by substitution with $u = \sec(x)$ and $du = \sec(x)\tan(x)$.**

   $$= \int \left(u^{3/2}-u^{-1/2}\right)\,du$$
   $$= \frac{2}{5}u^{5/2}-2u^{1/2}+C$$
   $$= \frac{2}{5}\sec^{5/2}(x)-2\sec^{1/2}(x)+C$$

Case 2: The power of secant is even and positive

Find $\int \sec^4(x)\tan^4(x)\,dx$.

1. **Lop off a $\sec^2(x)$ factor and move it to the right.**

$$=\int \sec^2(x)\tan^4(x)\sec^2(x)\,dx$$

2. **Convert the remaining secants to tangents with the Pythagorean identity, $\sec^2(x) = \tan^2(x) + 1$.**

$$=\int (\tan^2(x)+1)\tan^4(x)\sec^2(x)\,dx$$
$$=\int \left((\tan^6(x)+\tan^4(x)\right)\sec^2(x)\,dx$$

3. **Solve by substitution, where $u = \tan(x)$ and $du = \sec^2(x)$dx.**

$$=\int (u^6+u^4)\,du$$
$$=\frac{1}{7}u^7+\frac{1}{5}u^5+C$$
$$=\frac{1}{7}\tan^7(x)+\frac{1}{5}\tan^5(x)+C$$

Case 3: The power of tangent is even and positive and there are no secant factors

Integrate $\int \tan^6(x)\,dx$.

1. **Convert one $\tan^2(x)$ factor to secants using the Pythagorean identity, $\tan^2(x) = \sec^2(x) - 1$.**

$$=\int \tan^4(x)(\sec^2(x)-1)\,dx$$

2. **Distribute and split up the integral.**

$$=\int \tan^4(x)\sec^2(x)\,dx-\int \tan^4(x)\,dx$$

3. **Solve the first integral like in Step 3 of Case 2 for secants and tangents.**

   You should get $\int \tan^4(x)\sec^2(x)\,dx=\frac{1}{5}\tan^5(x)+C$

4. **For the second integral from Step 2, go back to Step 1 and repeat the process.**

   For this piece of the problem, you get

   $$-\int \tan^4(x)\,dx = -\int \tan^2(x)\sec^2(x)\,dx + \int \tan^2(x)\,dx$$

5. **Repeat Step 3 for $-\int \tan^2(x)\sec^2(x)\,dx$ (using Step 3 of Case 2 for secants and tangents again).**

   $$-\int \tan^2(x)\sec^2(x)\,dx = -\frac{1}{3}\tan^3(x) + C$$

6. **Use the Pythagorean identity to convert the $\int \tan^2(x)$ from Step 4 into $\int \sec^2(x)\,dx - \int 1\,dx$.**

   Both of these integrals can be done with simple reverse differentiation rules. After collecting all these pieces — piece 1 from Step 3, piece 2 from Step 5, and pieces 3 and 4 from Step 6 — your final answer should be $\int \tan^6(x)\,dx = \frac{1}{5}\tan^5(x) - \frac{1}{5}\tan^3(x) + \tan(x) - x + C$

Piece of cake.

Integrals containing cosecants and cotangents

Cosecant and cotangent integrals work exactly like the three cases for secants and tangents — you just use a different form of the Pythagorean identity: $1+\cot^2(x)=\csc(x)$. Try this one: Integrate $\int \frac{\cot^3(x)}{\sqrt{\csc(x)}}\,dx$. If you get $-2\sin^{1/2}(x) - \frac{2}{3}\csc^{3/2}(x) + C$, pass "Go" and collect $200.

TIP

There's more than one way to skin a cat. If you get a secant-tangent or a cosecant-cotangent problem that doesn't fit any of the cases discussed in the previous section or if you're otherwise stumped by a problem, try converting it to sines and cosines and solving it with one of the sine-cosine methods or with identities like $\sin^2(x)+\cos^2(x)=1$ and $\cos^2(x)=\frac{1+\cos(2x)}{2}$.

For example, $\int \frac{\tan^2(x)}{\sec^2(x)}\,dx$ doesn't fit any of the discussed cases, but you can convert it to $\int \sin^2(x)\,dx$ (using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$). You finish with the identity $\sin^2(x) = \frac{1-\cos\ (2x)}{2}$:

$$
\begin{aligned}
&\int \sin^2(x)\,dx \\
&= \int \frac{1-\cos\ (2x)}{2}\,dx \\
&= \int \frac{1}{2}\,dx - \int \frac{\cos\ (2x)}{2}\,dx \\
&= \frac{x}{2} - \frac{\sin(2x)}{4} + C
\end{aligned}
$$

You can also do many ordinary secant-tangent or a cosecant-cotangent problems by converting them into sine-cosine problems — instead of doing them the way described in this and the previous section.