# The Partial Fractions Technique Where the Denominator Contains Repeated Linear or Quadratic Factors

This article assumes you're familiar with the basics of the partial fractions technique for integration. If you've done some partial fractions problems, they likely involved integrals like $\int \frac{5}{(x-2)(x+3)}\,dx$ or $\int \frac{5x^3+9x-4}{x(x-1)(x^2+4)}\,dx$ where the denominators contained linear and/or irreducible quadratic factors with only one of each factor. To refresh your recollection, you'd begin the first integration problem by breaking up the fraction like this:

$$\frac{5}{(x-2)(x+3)} = \frac{A}{(x-2)} + \frac{B}{(x+3)}.$$

You'd break up the second integral like this:

$$\frac{5x^3+9x-4}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}.$$

(Notice how the numerator over the quadratic factor differs from the numerators over the linear factors.) The next step, as you probably know, is to solve for the capital-letter unknowns.

This article concerns the somewhat gnarlier problems where the denominators contain *repeated* linear or quadratic factors — that is, factors raised to a power more than 1. Say you want to integrate $\int \frac{1}{x^2(x-1)^3}\,dx$. Here's what you do. The $x$ in the denominator has a power of 2, so you get 2 partial fractions for the $x$ (for the powers of 1 and 2); the $(x-1)$ has a power of 3, so you get 3 partial fractions for that factor (for the powers 1, 2, and 3). Here's the general form for the partial fraction decomposition: $\frac{1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$.

Other than beginning the solution process with this longer type of decomposition, solving these problems is basically the same as solving the more basic problems without repeated factors. The algebra involved in solving for the capital-letter unknowns is a bit messier, but the approach is the same. Here's how you'd finish the current problem:

1. **Multiply both sides of the partial fractions equation by the left-side denominator.**

$$\frac{1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$
$$1 = A(x)(x-1)^3 + B(x-1)^3 + C(x^2)(x-1)^2 + D(x^2)(x-1) + Ex^2$$

2. **Take the roots of the linear factors (0 and 1) and plug them — one at a time — into *x* and solve.**

$$\begin{aligned} &\text{If } x = 0, && \text{If } x = 1, \\ &\quad 1 = B(-1)^3 && \quad 1 = E \cdot 1^2 \\ &\quad B = -1 && \quad E = 1 \end{aligned}$$

With these values for $B$ and $E$, now you've got

$$1 = A(x)(x-1)^3 - (x-1)^3 + C(x^2)(x-1)^2 + D(x^2)(x-1) + x^2$$

3. **Now plug into *x* values not used in Step 2 (low numbers make the arithmetic easier) to get a system of equations in *A*, *C*, and *D*. Let's start with –1 and 2.**

If $x = -1$,
$$\begin{aligned} 1 &= 8A + 8 + 4C - 2D + 1 \\ -8 &= 8A + 4C - 2D \\ -4 &= 4A + 2C - D \end{aligned}$$

If $x = 2$,
$$\begin{aligned} 1 &= 2A - 1 + 4C + 4D + 4 \\ -2 &= 2A + 4C + 4D \\ -1 &= A + 2C + 2D \end{aligned}$$

You've got three unknowns, $A$, $C$, and $D$, so you need a system of three equations, right? And you could easily get a third equation by just using a third value for $x$, say, $x = 3$. But let's get the third equation with a different trick which you should have in your calculus toolbox.

4. **Equate the coefficients of the $x^4$ term on the left and right sides of the equation.**

   Look at the equation at the bottom of Step 2. Can you see that if you were to expand the right side of the equation, the only $x^4$s you'd get would be $Ax^4 + Cx^4$**?** Thus, on the right side of the equation, the coefficient of $x^4$ would be $(A + C)$. Since there are no $x^4$s on the left side of the equation, in other words, the coefficient of $x^4$ on the left side is zero, that tells you that $A + C = 0$. And that's your third equation. Piece o' cake, right?

5. **Solve your system of three equations in three unknowns (the two equations from Step 3 and the single equation from Step 4).**

   You're on your own for this basic algebra problem. You should get the following: $A = -3$, $C = 3$, and $D = -2$. This result plus the values for $B$ and $E$ from Step 2 give you the solved partial fractions decomposition:

   $$\frac{1}{x^2(x-1)^3} = -\frac{3}{x} - \frac{1}{x^2} + \frac{3}{(x-1)} - \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$$

   Thus,

   $$\int \frac{1}{x^2(x-1)^3}\,dx = -\int \frac{3}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{3}{(x-1)}\,dx - \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx$$

6. **Do the final integration.**

   These integrals should give you no trouble. For the first and third, you use the natural log rule, and for the other three you use the reverse power rule. Here's your final answer:

   $$\begin{aligned}\int \frac{1}{x^2(x-1)^3}\,dx &= -3\ln|x| + \frac{1}{x} + 3\ln|x-1| + \frac{2}{(x-1)} - \frac{1}{2(x-1)^2} + C\\ &= 3\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + \frac{2}{(x-1)} - \frac{1}{2(x-1)^2} + C\end{aligned}$$

You might see an even nastier problem where there are repeated linear and quadratic factors in the fraction's denominator. Say you're asked to integrate $\int \frac{4x^3 - x^2 + 8}{(2x-3)^2(x^2+1)^2}\,dx$. Here's how you'd do the partial fractions decomposition. You'd break up $\frac{4x^3 - x^2 + 8}{(2x-3)^2(x^2+1)^2}$ into $\frac{A}{(2x-3)} + \frac{B}{(2x-3)^2} + \frac{Cx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2}$.

Notice that, similar to the first problem, you get two fractions for each of the two factors in the original denominator (one containing the factor raised to the first power and one containing the factor raised to the second power). Like with all partial fractions problems, your next task is to solve for the capital-letter unknowns. Don't bother trying to finish this one, however. It's *way* too messy.