Finding Volume with the Matryoshka Doll Method (a.k.a. the Cylindrical Shell Method)

If you've studied some integral calculus, you've no doubt run across the disk and washer methods for computing volume. Those methods involve cutting up a given solid into thin slices, each the shape of a circular disk or washer (a disk with a hole in its middle). You then add up (integrate) the volumes of all the slices to get the total volume.

With the cylindrical shell method, you instead cut up the given solid into thin concentric cylinders and then add up the volumes of all the cylinders. The concentric cylinders fit inside each other kinda like those nested Russian dolls. Can you picture how they fit inside each other? Imagine a soup can that somehow has many paper labels, each one covering the one beneath it. Or picture one of those de-linter rolls with the sticky papers you peel off. Each soup can label or piece of sticky paper is a cylindrical shell — before you tear it off, of course. After you tear it off, it's an ordinary rectangle. Because each unrolled cylinder becomes a rectangle, the formula for the area of a rectangle is the key to the cylindrical shell method.

Here's a problem: A solid is generated by taking the area bounded by the x-axis, the lines $x = 2$, $x = 3$, and $y = e^x$, and then revolving that area about the y-axis. See Figure 1.

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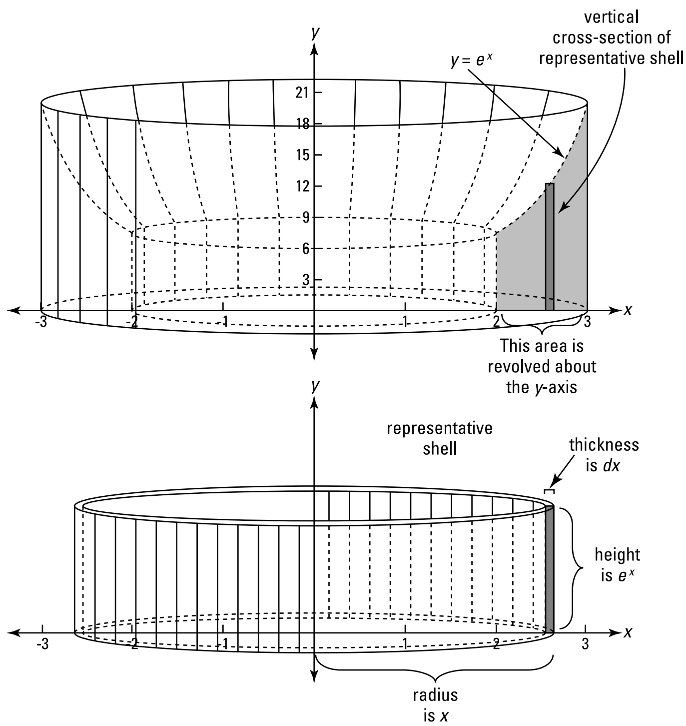

Figure 1: A shape sort of like the Roman coliseum and one of its representative shells.

What's the volume?

1. **Determine the area of a representative cylindrical shell.**

   TIP

   When picturing a representative shell, focus on a shell that's in no place in particular. Figure 1 shows such a generic shell. Its radius is unknown, x, and its height is the height of the curve at x, namely e^x. If, instead, you use a special shell like the outer-most shell with a radius of 3, you're more likely to make the mistake of thinking that a representative shell has some *known* radius like 3 or a *known* height like e^3. Both the radius and the height are *unknown.*

   Each representative shell, like a soup can label or a sticky sheet from a de-linter, is just a rectangle whose area is, of course, *length* times *width.* The rectangular soup can label goes all the way around the can, so its length is the circumference of the can, namely $2\pi r$; the width of the label is the height of the can. So now you've got the general formula for the area of a representative shell:

$$\begin{aligned} Area &= length \cdot width \\ &= 2\pi r \cdot h \end{aligned}$$

For the current problem, you plug in x for the radius and e^x for the height, giving you the area of a representative shell:

$$\textit{Area of shell} = 2\pi x \cdot e^x$$

2. **Multiply the area by the thickness of the shell, *dx*, to get its volume.**

$$\textit{Volume of representative shell} = 2\pi x e^x \cdot dx$$

3. **Add up the volumes of all the shells from 2 to 3 by integrating.**

$$\begin{aligned}
\textit{Total volume} &= \int_2^3 2\pi x e^x\, dx \\
&= 2\pi \int_2^3 x e^x\, dx \\
&= 2\pi\left[x e^x - e^x\right]_2^3 \qquad \text{(integration by parts)} \\
&= 2\pi\left((3e^3 - e^3) - (2e^2 - e^2)\right) \\
&= 2\pi(2e^3 - e^2) \\
&\approx 206 \text{ cubic units}
\end{aligned}$$

TIP

With the disk and washer methods, it's usually pretty obvious what the limits of integration should be (recall that the *limits of integration* are, for example, the 1 and 5 in $\int_1^5$). With cylindrical shells, however, it's not always quite as clear. Here's a tip: You integrate from the *right* edge of the smallest cylinder to the *right* edge of the largest cylinder (like from 2 to 3 in the problem above). And note that you *never* integrate from the left edge to the right edge of the largest cylinder (like from –3 to 3).